Inertial Forces as Viewed from the ADM Slicing and Their Behaviour for Particles in Non-Circular Geodesics

A.R.Prasanna

Physical Research Laboratory

Ahmedabad 380 009, India

(email: prasanna@prl.ernet.in)

Abstract

Considering the definition of inertial forces acting on a test particle, following non-circular geodesics, in static and stationary space times we show that the centrifugal force reversal occurs only in the case of particles following prograde orbits around black holes. We first rewrite the covariant expressions for the acceleration components in terms of the lapse function, shift vector and the 3-metric γ_{ij}, using the ADM 3+1 splitting and use these, for different cases as given by pure radial motion, pure azimuthal motion and the general non-circular motion. It is found that the reversal occurs only when the azimuthal angular velocity of the particle supersedes the radial velocity, which indeed depends upon the physical parameters E, ℓ and the Kerr parameter a.

PACS Numbers: 0470B, 0420, 9530S, 9760L

Keywords: inertial forces, geodesics

1 Introduction

Though the idea of introducing the concept of inertial forces within the framework of general relativity is more than a decade old (Abramowicz et al., 1988, 1990, 1993), one of the significant results of doing so, viz. the reversal of centrifugal force (Abramowicz and Prasanna, 1990) seems to have found some applicability in the context of astrophysical scenario recently. Heyl (2000) while probing the properties of neutron stars with type 1 X-ray bursts, concludes that, if the change in spin frequency is due to a change in the thickness of the atmosphere, the radius of the star must exceed $3m$ for any equation of state, the constraint arising from the reversal of centrifugal force. He also considers the effect of the Coriolis type of force arising from the dragging of inertial frames and its relevance in the context of calculating the angular momentum of the fluid element in the neutron star atmosphere. More recently, Hasse and Perlick (2001) while discussing gravitational lensing in spherically symmetric static space times with centrifugal force reversal, show that in any spherically symmetric static space time, the upper limiting radius for a non-transparent Schwarzschild source to act as a gravitational lens that produces infinitely many images, is intimately related to the radius at which the centrifugal reversal takes place. While these studies are encouraging to show the relevance of the introduction of 'inertial forces' in general relativity, one has to bear in mind that almost all the results obtained earlier in the context of 'centrifugal reversal' were for the study of test particles in circular geodesics. In fact, Gupta, Iyer and Prasanna (1996) had obtained the expression for the centrifugal force on a fluid element, in the optical reference geometry approach while considering the behaviour of ellipticity of a slowly rotating configuration which is ultra compact, but the treatment was not fully covariant.

We shall now set up a formalism using the covariant prescription for inertial forces (Abramowicz et al. (1995) and relating it to the study of test fluids on a given background geometry in the ADM splitting with shift vector and lapse function and rewrite the components of acceleration in terms of these quantities and the velocity 3-vector V^i. With this formalism, one can perhaps look for the behaviour of the inertial forces in general dynamical space times using numerical techniques.

In the present paper, we shall use the formalism for considering fluid flow in stationary space times with only radial and only azimuthal components of velocity being non-zero and then the case of static space time wherein both the radial and azimuthal components of particle three velocity are non-zero (non-circular godesics).

2 Formalism

On a general manifold M described by the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1}$$

One can use the 3+1 ADM splitting to define the lapse function α, shift vector β^i and the 3-metric γ_{ij} (York 1983) to rewrite (1) as

$$ds^2 = -\left(\alpha^2 - \beta_i \beta^i\right) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^i \tag{2}$$

It is well known that this splitting forms the basis for numerical hydrodynamics, wherein with the introduction of a unit time-like vector field n^μ normal to the space-like hypersurface $\sum$ (t = const), one can define the fluid 3-velocity V^i (York 1983)

$$V^i = \frac{U^i}{\alpha U^t} + \frac{\beta^i}{\alpha} \tag{3}$$

with U^i being the spatial component of the fluid four velocity $U^\mu = (U^t, U^i)$.

In order to introduce the "inertial forces" (Newtonian) one can follow the covariant procedure as given by Abramowicz (1995), and relate the time-like vector field n^μ to the gradient of the gravitational potential ϕ through

$$n^\nu \nabla_\nu n_\mu = -\nabla_\mu \Phi \tag{4}$$

with

$$\phi = \frac{1}{2}\ell n \left[- < \eta, \eta > -2\omega < \xi, \eta > -\omega^2 < \xi\xi >\right] \tag{5}$$

for any stationary axisymmetric space-time with η and ξ denoting the time-like and space-like Killing Vectors.

If τ^μ represents a space-like vector field, orthogonal to n^μ, then the four velocity in the space-time may be written as

$$\begin{aligned} U^\mu &= \gamma \left(n^\mu + V\tau^\mu\right) \\ \gamma &= n^\mu U_\mu = \tfrac{1}{\sqrt{1-V^2}} \end{aligned} \tag{6}$$

and the three metric

$$\begin{aligned} \gamma_{ij} &= g_{ij} + n_i n_j \\ \gamma_{ij} V^i V^j &= V^2 \end{aligned} \tag{7}$$

If we now introduce the conformal rescaling of the 3+1 splitting as done by Abramowicz, Carter and Lasota (1988) using what is called the Optical Reference Geometry

$$\begin{aligned} \tilde{\gamma}_{ik} &= e^{2\phi}\gamma_{ik} \\ \tilde{\tau}^i &= e^{-\phi}\tau^i \end{aligned} \tag{8}$$

then the four acceleration $a_\mu = U^\nu \nabla_\nu U_\mu$ can be written in the explicit form (Abramowicz 1993)

$$\begin{aligned} a_\mu : &= -\nabla_\mu \phi + \gamma^2 V \left(n^\nu \nabla_\nu \tau_\mu + \tau^\nu \nabla_\nu n_\mu\right) + \dot{\gamma} n_\mu \\ &+ \tilde{V}^2 \tilde{\tau}^\nu \tilde{\nabla}_\nu \tilde{\tau}_\mu + \left(\dot{E} V e^\phi + \gamma \dot{V}\right)\tau_\mu \end{aligned} \tag{9}$$

wherein the terms can get the specific identifications as gravitatonal, Coriolis type (Lense-Thirring) $\theta(V)$, centrifugal $\theta\left(V^2\right)$ and Eulerian accelerations. $\tilde{\nabla}_\nu$ is the covariant derivative in the absolute 3-space with the projected conformal metric $\tilde{\gamma}_{ij}$.

One can now relate these two splittings of the axisymmetric, stationary space-time through the definition

$$\begin{aligned} n^\mu &= \left(\tfrac{1}{\alpha}, -\tfrac{\beta^i}{\alpha}\right) \\ n_\mu &= (-\alpha, 0) \\ \tau^\mu &= \left(0, \tfrac{V^i}{V}\right) \\ \tau_\mu &= \left(\tfrac{\beta^i V_i}{V}, \tfrac{V_i}{V}\right) \end{aligned} \tag{10}$$

Using (9) and (10) one can express the components of acceleration vector a_μ in terms of α, β^i, γ_{ij} and the 3-velocity components V^i.

The centrifugal acceleration acting on a fluid element $\tilde{V}^2\tilde{\tau}^\nu\tilde{\nabla}_\nu\tilde{\tau}_\mu$ is given by

$$(Cf)_i = \gamma^2\left[VV^j\partial_j\left(\frac{V_i}{V}\right) + \left(V_iV^j\partial_j - V^2\partial_i\right)(\Phi) - \frac{1}{2}V^jV^k\partial_i\gamma_{jk}\right] \tag{11}$$

while the Coriolis type (Lense-Thirring) is given by

$$(Co)_i = -\frac{\gamma^2}{\alpha}\left[V\beta^j\partial_j\left(\frac{V_i}{V}\right) + V^j\partial_i g_{oj} - \beta^kV^j\partial_i\gamma_{kj}\right] \tag{12}$$

Thus for any given background geometry one can evaluate the specific acceleration components, if one has the 3-velocity field of the fluid on that geometry evaluated through the equations of motion.

3 Specific Examples

3.1 Purely Radial Flow $\left(V^r \neq 0, V^\phi = 0, V^\theta = 0\right)$

A purely radial flow which is encountered in the case of spherical accretion, particularly in static space times will have

$$(Cf)_\gamma = \gamma^2 \left[V V^r \partial_r \left(\frac{V_r}{V} \right) - \frac{1}{2} \left(V^r \right)^2 \partial_r \gamma_{rr} \right] \tag{13}$$

As $V^2 = \gamma_{rr} \left(V^r \right)^2$ and $V_r = \gamma_{rr} V^r$, the first term also gives $\frac{1}{2} \left(V^r \right)^2 \partial_r \gamma_{rr}$ and thus $(Cf)_r$ is identically zero. This is to be expected and goes to show the consistency of the definition.

3.2 Purely Azimuthal Flow $\left(V^r = 0, V^\theta = 0, V^\phi \neq 0\right)$

$$\begin{aligned}
(Cf)_r &= \gamma^2 \left[V V^r \partial_r \left(\tfrac{V_r}{V} \right) + \left(V^r V_r \partial_r - V^2 \partial_r \right) (\Phi) - \tfrac{1}{2} V^j V^k \partial_r \gamma_{jk} \right] \\
&= \gamma^2 \left[-\gamma_{\phi\phi} \left(V^\phi \right)^2 \left(\partial_r \Phi \right) - \tfrac{\left(V^\phi \right)^2}{2} \partial_r \gamma_{\phi\phi} \right] \\
(Cf)_\gamma &= -\tfrac{\gamma^2 \left(V^\phi \right)^2}{2} \left[2\gamma_{rr} \left(\partial_r \Phi \right) + \partial_r \gamma_{\phi\phi} \right]
\end{aligned} \tag{14}$$

a. Schwarzschild Space Time:

$$\begin{aligned}
\Phi &= -\tfrac{1}{2} ln \left(1 - \tfrac{2m}{r} \right) \\
\gamma_{\phi\phi} &= r^2 \\
(Cf)_r &= -\gamma^2 \left(V^\phi \right)^2 r \left(1 - \tfrac{2m}{r} \right)^{-1} \left(1 - \tfrac{3m}{r} \right)
\end{aligned} \tag{15}$$

b. Kerr Space Time:

$$\begin{aligned}
\Phi &= -\tfrac{1}{2} ln \left\{ \frac{\left(r^3 + a^2 r - 2mr^2\right)}{\left(r^3 + a^2 r + 2ma^2\right)} \right\} \\
\gamma_{\phi\phi} &= r^2 + a^2 + \tfrac{2ma^2}{r} \\
\Delta &= r^2 - 2mr + a^2 \\
(Cf)_r &= \tfrac{\gamma^2 \left(V^\phi\right)^2}{r^2 \Delta} \left(r^5 - 3mr^4 + a^2 \left(r^3 - 3mr^2 + 6m^2 r - 2m^2\right)\right)
\end{aligned} \tag{16}$$

The results in this case is again as expected and is the same as in the test particle case (Abramowicz and Prasanna (1990), Iyer and Prasanna (1993)).

Case (3): $V^r \neq 0$, $V^\phi \neq 0$, $V^\theta = 0$.

(a) As mentioned earlier, the discussion of inertial forces was earlier confined to particles in circular motion only. Hence, it would be useful first to consider the case of a test particle with both radial and azimuthal velocity non-zero. We shall restrict the discussion for static geometry as given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{17}$$

For a test particle following a geodesic, confined to the equatorial plane $\theta = \frac{\pi}{2}$, the equations of motion reduce to

$$\left(1 - \frac{2m}{r}\right) \frac{dt}{ds} = E \tag{18}$$

$$r^2 \frac{d\phi}{ds} = \ell \tag{19}$$

and

$$\frac{dr}{ds} = \sqrt{E^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)} \tag{20}$$

From the definition of the 3-velocity $V^i = \frac{u^i}{\alpha u^o}$, we then get

$$(V^r)^2 = \left(1 - \frac{2m}{r}\right)\left[1 - \frac{1}{E^2}\left(1 - \frac{2m}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right] \tag{21}$$

and

$$\left(V^\phi\right)^2 = \frac{\ell^2}{E^2 r^4}\left(1 - \frac{2m}{r}\right) \tag{22}$$

Using these in the expression for the centrifugal force, one finds

$$Cfr = \frac{-m\ell^2}{r^3\left[2m - \left(1 - E^2\right) r\right]} \tag{23}$$

which clearly shows no reversal for any value of E and ℓ, outside the horizon.

(b) We shall consider a second example of static spacetime, but with an additional interacting field namely that of a charged particle in the Reissner-Nordstrom geometry.

$$\begin{aligned} ds^2 &= -\left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right) dt^2 - \left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right)^{-1} dr^2 \\ &\quad -r^2 d\theta^2 - r\sin^2\theta d\phi^2 \end{aligned}$$

For a charged test particle with charge e one has the energy E and angular momentum ℓ as given by

$$\begin{aligned} U^t &= \left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right)^{-1}\left(E - \tfrac{eQ}{r}\right) \\ U^\phi &= \tfrac{\ell}{r^2} \end{aligned}$$

which in turn give for the components of three velocity

$$\begin{aligned} (V^r)^2 &= \left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right)\left\{1 - \left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right)\left(1 + \tfrac{\ell^2}{r^2}\right) / \left(E - \tfrac{eQ}{r}\right)^2\right\} \\ \left(V^\phi\right)^2 &= \tfrac{\ell^2}{r^4}\left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2}\right) / \left(E - \tfrac{eQ}{r}\right)^2 \end{aligned}$$

Evaluating the centrifugal force acting on such a particle one finds

$$(Cfr) = \frac{\ell^2\left[\left(1 - e^2\right) Q^2 - mr + eEQr\right]}{r^3\left\{\left(1 - e^2\right) Q^2 - 2eEQr + r\left(2m - \left(1 - e^2\right) r\right)\right\}}$$

It is easy to see that Cfr can be zero only for $r = Q^2\left(1 - e^2\right) / (m - eEQ)$, which is $< m$ and thus inside the event horizon.

(c) Stationary Space-time:

We consider the case of a test particle in a non-circular orbit (V^r, V^ϕ, both non-zero) on the equatorial plane of the Kerr geometry which represents the space-time outside a black hole as given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{r^2}{\Delta} dr^2 + \frac{4am}{r} d\phi dt + \left(r^2 + a^2 + \frac{2ma^2}{r}\right) d\phi^2$$

The constants of motion E and ℓ of the particle are now related as given by

$$\begin{aligned} U^\phi &= \tfrac{d\phi}{ds} = \tfrac{1}{\Delta}\left[\left(1 - \tfrac{2m}{r}\right)\ell + \tfrac{2am}{r}E\right] \\ U^t &= \tfrac{dt}{ds} = \tfrac{1}{\Delta}\left[\left(r^2 + a^2 + \tfrac{2ma^2}{r}\right)E - \tfrac{2ma}{r}\ell\right] \end{aligned}$$

and the radial four velocity U^r is given by

$$(U^r)^2 = \left(\frac{dr}{ds}\right)^2 = E^2 + \frac{2m}{r^3}(aE - \ell)^2 + \frac{1}{r^2}\left(a^2E^2 - \ell^2\right) - \frac{\Delta}{r^2}$$

The lapse function α, the shift vector β^i and the 3-metric γ_{ij} are given by

$$\begin{aligned} \alpha^2 &= \frac{\Delta}{\left(r^2 + a^2 + \frac{2ma^2}{r}\right)} \\ \beta^i &= \left(0,\ 0,\ \frac{-2am}{(r^3 + a^2 r + 2ma^2)}\right) \end{aligned}$$

and

$$\gamma_{ij} = \begin{bmatrix} \frac{r^2}{\Delta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(r^2 + a^2 + \frac{2ma^2}{r}\right) \end{bmatrix}$$

From these expressions one can easily obtain the radial and azimuthal velocities of the particle in the 3-space as given by

$$\begin{aligned} V^r &= \frac{U^r}{\alpha U^t} \\ V^\phi &= \frac{\left(U^\phi/U^t + \beta^\phi\right)}{\alpha} \end{aligned}$$

Using these in the expression for the 'Centrifugal Acceleration' $(Cf)r$

as given by (11), one finds

$$(Cf)_r = \frac{\begin{array}{l} -\left[\ell^2 mr\left\{r^7 - 6aE\ell r^5 + (r+2m)\left(a^6 - 2a^5E\ell\right)\right.\right. \\ -4a^3E\ell r^2(3m+2r) + a^4\left(4m\ell^2 - 8m^2r + 3r^3\right) \\ \left.\left.+a^2\left(12\ell^2mr^2 + r^4(3r-2m)\right)\right\}\right] \end{array}}{\begin{array}{l} \left[\left(r^3 + a^2(2m+r)\right)^2\left(-4aE\ell mr^3 - 4a^3E\ell m(r+2m) + \right.\right. \\ r^5\left(2m + \left(E^2-1\right)r\right) + a^4(r+2m)\left(E^2(r+2m) - r\right) \\ \left.\left.+2a^2\left(2\ell^2m^2 + r^2\left(2m^2 + 2E^2mr + \left(E^2-1\right)r^2\right)\right)\right)\right] \end{array}}$$

If one looks for solutions of the equation for $(Cf)_r = 0$, in principle, one finds eight roots of which there may be just one or two outside the horizon $r_+ = m + \sqrt{m^2 - a^2}$, for different combinations of E and ℓ. However, if one also demands the physicality of the root by looking at the condition $(U^r)^2 > 0$, then it appears that there exists one root between $2m < r < 3m$, and the exact location depends upon the combination of E and ℓ of the particle. One can also see that the location of the realistic root for $(Cf)_r = 0$ appears after the point wherein $\frac{U^\phi}{U^t}$ exceeds $\frac{U^r}{U^t}$ for the particle moving inwards from infinity.

4 Discussion

The reversal of centrifugal force acting on a test particle in circular orbit at the last circular unstable photon orbit, in static space time has drawn some attention in recent times in varying context ranging from X-ray sources to infinitely multiple image forming. Actually one can clearly see that these features are closely associated with the photon behaviour in static space times. Though test particle trajectories do give some understanding of the geometry on which they are moving, for astrophysical applications it is fluid flows that are important.

Having obtained the expressions for the inertial accelerations in terms of the fluid velocity components and the metric components one can now

write down the individual forces on any background space time, provided one has the solution for the velocity components on the given background for the flow. For a dusty fluid ($p = 0$) as may be expected for a purely radial flow $\left(V^{\theta} = 0, V^{\phi} = 0\right)$ the centrifugal acceleration is zero, whereas for a purely azimuthal flow $\left(V^{r} = 0, V^{\theta} = 0, V^{\phi} \neq 0\right)$, the centrifugal acceleration is just as for a test particle, both in Schwarzschild and the Kerr background. The fluid in this case is a collection of test particles in circular orbits and thus the result is as known earlier.

However, it is important to understand the behaviour of forces for a more general motion of particles, when the trajectory is a non-circular geodesic. For such a case, it is easy to find the expressions for three velocity components from the components of the four velocity, which are directly integrable from the equations of motion using the symmetries of the given space time. This would bring in the physical characteristics of the particle, the energy E and angular momentum ℓ and restricting the discussion to particles on the equatorial plane $\left(\theta = \frac{\pi}{2},\ U^{\theta} = 0\right)$, one can easily obtain the components V^{r} and V^{ϕ} for the particle.

From the expression for $\left(Cf\right)_{r}$ in Kerr space time, it is clear that its behaviour depends upon the particle parameters E and ℓ and the rotation parameter 'a'. Direct evaluation for fixed E and ℓ shows that when $E \leq 1$, the reversal radius occurs outside the event horizon only for very high value of a (Table 1). As may be seen, the table also gives for the same set of parameters the radius at which $\left(F_3 = \frac{U^{\phi}}{U^{t}}\right)$ crosses over $\left(F_2 = \frac{U^{r}}{U^{t}}\right)$, and it is clear that the centrifugal reversal occurs only after the angular velocity supersedes the radial velocity. Figures 1 to 3 show the behaviour of the curves, centrifugal $(f1)$, radial velocity $(f2)$ and the azimuthal velocity $(f3)$

for different values of a, for same E and ℓ.

Indeed one can see that the centrifugal reversal which was inherent for circular geodesics in both static and stationary spacetimes, does not follow automatically for general non-circular geodesics. Whereas it does not occur at all in static spacetime, in stationary spacetime, the occurrence depends upon the energy, angular momentum of the particle and the rotation parameter a, which needs to be sufficiently high for the reversal to occur outside the event horizon. The difference in the behaviour for static and stationary spacetimes is essentially due to the fact that the rotation induces ‘frame dragging’, which adds to the azimuthal velocity and makes it larger than the radial velocity after which the behaviour resembles that of a circular orbit. This interpretation gets further support from the fact that for particles in retrograde motion $(a > 0,\ \ell < 0)$, there are no positive real roots for the equation $(Cf)_r = 0$.

Thus we find that for particles on non-circular geodesics, in static spacetime, there is no reversal of centrifugal force, and in Kerr spacetime, the reversal occurs only for particles in prograde motion, which get the additional input to their angular velocity from the effects of frame dragging. For particles in retrograde motion, the frame dragging contribution would not suffice to overcome the effects of radial velocity and thus like in static case shows no reversal.

Table 1

Location of the last root of $CFr = 0$ and the second root of $(F2 - F3) = 0$, alongwith the location of event horizon (EH) for different values of a

a	EH	$(Cfr) = 0$	$(F2 - F3) = 0$
0.1	1.99499	—-	2.0615
0.2	1.9798	—-	2.10945
0.3	1.95394	—-	2.14586
0.4	1.91652	1.28011	2.17201
0.5	1.86603	1.48094	2.18871
0.6	1.8	1.54586	2.19647
0.7	1.71414	1.78834	2.19554
0.8	1.6	1.91397	2.18595
0.9	1.43589	2.02589	2.16752
1.0	1.0	2.12612	2.13987

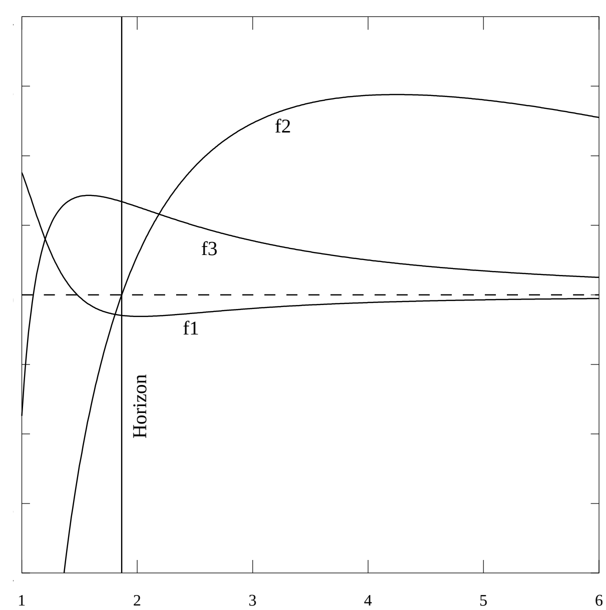

Fig. 1: Plots of $Cfr\,(f1)$, $\frac{U^r}{U^t}\,(f2)$ and $\frac{U^\phi}{U^t}\,(f3)$ for the parameter values $E = 0.9$, $\ell = 1$ and $a = 0.5$. The vertical line shows the location of the horizon.

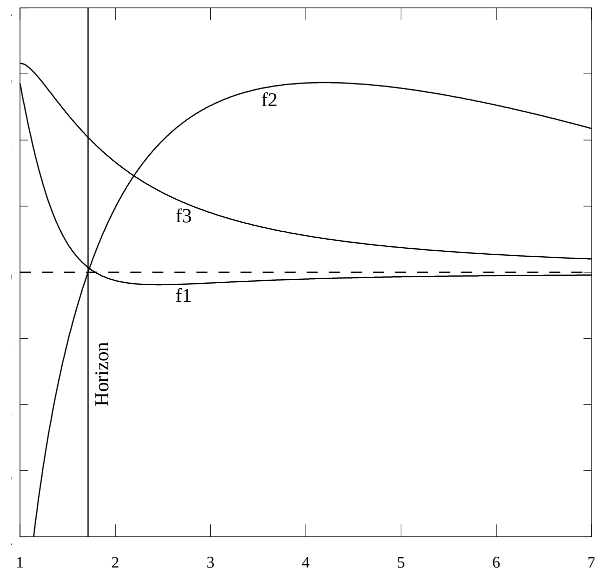

Fig. 2: Same as Fig. 1, but for $a = 0.7$.

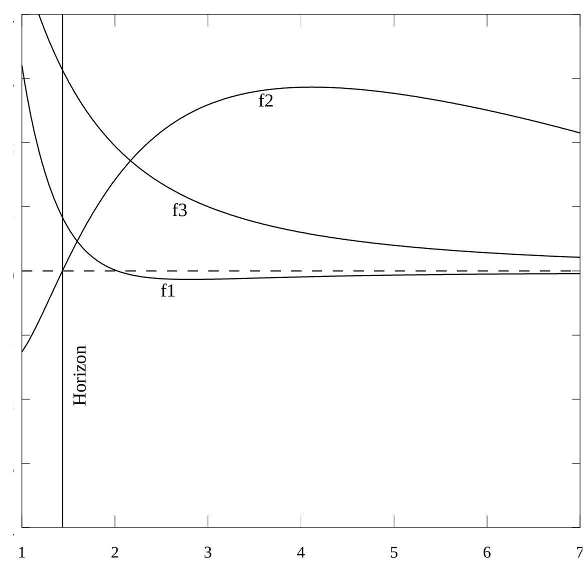

Fig. 3: Same as Fig. 1, but for $a = 0.9$.

It is well known that the Kerr parameter 'a' in the stationary case is interpreted through the asymptotic boundary condition, as defining the rotation of the source with respect to distant fixed frame. The concepts of inertial forces, the centrifugal and Coriolis, as known, do depend on the existence of a far away inertial frame in a fixed background. The inertial frame dragging as envisaged in the Kerr geometry is indeed considered as

a true Machian effect (Brill (1994)). We have seen above how this effect influences the particle in curved geometry. As the 3-velocity of the test particle does depend upon the local physics, one can clearly see from the above example the inherent aspects of Mach's principle in general relativity.

Acknowledgement

It is a pleasure to thank Ewald Muller, Toni Font, John Miller and Marek Abramowicz for discussions at various stages of this work and Subharti Ray for the help provided in making the plots.

References

Abramowicz M.A., Carter B. and Lasota, J.P. (1988), *Gen. Rel. & Grav.* **20**, 1173.

Abramowcicz M.A. and Prasanna A.R. (1990) *Mon. Not. R. Astr. Soc.* **245**, 720.

Abramowicz M.A. (1993) *Sci. Am.* **268**, 74.

Abramowicz M.A., Nurowski P. and Wex N. (1993) *Class. & Qu. Gr.* **10**, L183.

Brill D.R. (1993) *Mach's Principle*, Proceedings of the Tübingen Conference (1993), Einstein Studies Vol. 6, Eds. J. Barbour and H. Pfister, pp. 208-213, Birkhaüser, Boston.

Gupta A., Iyer S. and Prasanna A.R. (1996) *Class. & Qu. Gr.* **13**, 2675.

Iyer S. and Prasanna A.R. (1993) *Class. & Qu. Gr.* **10**, L13.

Hasse W. and Perlick V. (2001) *gr-qc* 0108002.

Heyl J.S. (2000) *Ap. J. (Letters)* **542**, L45.

York J.W. Jr. (1983) '*Gravitational Radiation*', ed. N. Deruelle and T. Piran (North Holland) 175.